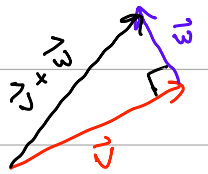
Keep in mind this is the continuation of last lectures notes

Def - The length of $\vec{v} \in \mathbb{R}^n$ is $\|\vec{v}\| = \sqrt{\vec{v} \cdot \vec{v}}$

Def - Two vectors $\vec{v}, \vec{w} \in \mathbb{R}^n$ are <u>orthogonal</u> if $\vec{v} \cdot \vec{w} = 0$.

$\boxed{\text{Example}}$ $\begin{pmatrix} 4 \\ -1 \\ 2 \end{pmatrix}$ + $\begin{pmatrix} 1 \\ 2 \\ -1 \end{pmatrix}$ are orthogonal

<u>Motivation</u> $\vec{v}$ + $\vec{w}$ are $\perp$ if

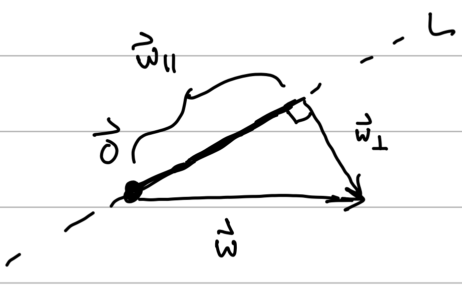


= right Δ i.f.f. $\|\vec{v}\|^2 + \|w\|^2 = \|v+w\|^2$

$$\vec{v} \cdot \vec{v} + \vec{w} \cdot \vec{w} \overset{??}{=} (\vec{v}+\vec{w}) \cdot (\vec{v}+\vec{w})$$

$$\cancel{\vec{v} \cdot \vec{v}} + \cancel{\vec{w} \cdot \vec{w}} = \cancel{\vec{v} \cdot \vec{v}} + \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{v} + \cancel{\vec{w} \cdot \vec{w}}$$

$$0 = \vec{v} \cdot \vec{w} + \vec{w} \cdot \vec{v} = 2 \vec{v} \cdot \vec{w}$$



Proof : Let $\vec{w}_{\parallel} = c\vec{v}$ for $c \in \mathbb{R}$ TBD

Then $\vec{w} = \vec{w}_{\parallel} + \vec{w}_{\perp} \Leftrightarrow \vec{w}_{\perp} = \vec{w}_{\perp} = \vec{w} - c\vec{v}$

And $0 = \vec{w}_{\perp} \cdot \vec{v} = (\vec{w} - c\vec{v}) \cdot \vec{v} = \vec{w} \cdot \vec{v} - c\,\vec{v} \cdot \vec{v}$

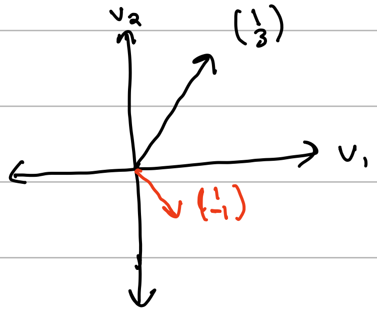
$\Longleftrightarrow$ $c = \dfrac{\vec{w} \cdot \vec{v}}{\vec{v} \cdot \vec{v}}$

so prop is true for $\left[\vec{w}_{\parallel} = \left(\dfrac{\vec{w} \cdot \vec{v}}{\vec{v} \cdot \vec{v}} \right) \vec{v} \right]$

$\boxed{\text{Example}}$ Orthogonal Projection

e.g. $\vec{v} = (1,3)$

$\vec{w} = (1,-1)$



$$\vec{w}_{\parallel} = \frac{\begin{pmatrix}1\\-1\end{pmatrix}\cdot\begin{pmatrix}1\\3\end{pmatrix}}{\begin{pmatrix}1\\3\end{pmatrix}\cdot\begin{pmatrix}1\\3\end{pmatrix}}\begin{pmatrix}1\\3\end{pmatrix} = \frac{(1-3)}{10}\begin{pmatrix}1\\3\end{pmatrix} = \begin{pmatrix}-1/5\\3/5\end{pmatrix}$$

<u>Def</u> - Let $\vec{v} \in \mathbb{R}^n$ be a non-$\vec{0}$ vector and L be the line through $\vec{0} + \vec{v}$.
Then the <u>orthogonal</u> <u>projection</u> of $\vec{w} \in \mathbb{R}^n$ onto L is the vector

<u>Prop</u>: $\text{proj}_L(\vec{w}) : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation

Proof (one property):

Check for $c \in \mathbb{R}$, $\vec{x} \in \mathbb{R}^n$

$$T(c\vec{x}) = \frac{c\vec{x}\cdot\vec{v}}{\vec{v}\cdot\vec{v}}\vec{v} = cT(\vec{x})$$

Example

$$\vec{v} = \begin{pmatrix} 1 \\ 3 \end{pmatrix}$$

$$\frac{\begin{pmatrix} 0 \\ 1 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix}}{\begin{pmatrix} 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix}} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 3/10 \\ 9/10 \end{pmatrix}$$

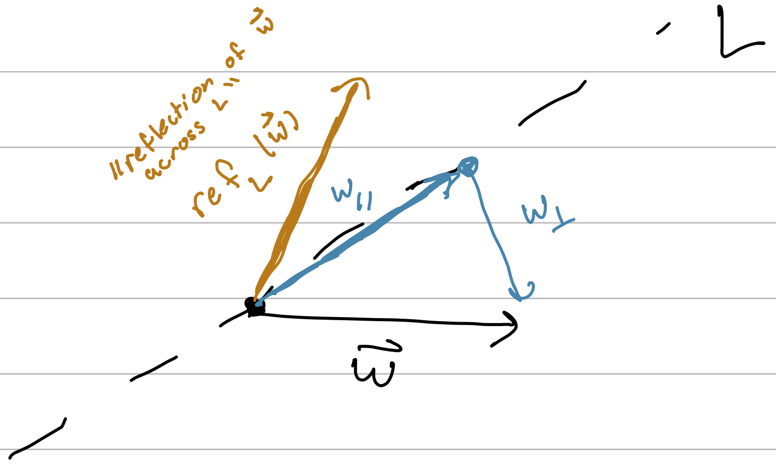
$$\left[\text{proj}_L \right] = \left[\text{proj}_L(\vec{e}_1) \quad \text{proj}_L(\vec{e}_2) \right]$$

$$\frac{\begin{pmatrix} 1 \\ 0 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix}}{\begin{pmatrix} 1 \\ 3 \end{pmatrix} \cdot \begin{pmatrix} 1 \\ 3 \end{pmatrix}} \begin{pmatrix} 1 \\ 3 \end{pmatrix} = \begin{pmatrix} 1/10 \\ 3/10 \end{pmatrix}$$

first column of matrix

$$\left[\text{proj}_L \right] = \begin{bmatrix} 1/10 & 3/10 \\ 3/10 & 9/10 \end{bmatrix}$$

- So now you can multiply any vector by the matrix $\left[\text{proj}_L \right]$ to get the output vector instead of using the formula.



<u>Def</u> - The reflection $\text{ref}_L(\vec{w})$ of a vector $\vec{w} \in \mathbb{R}^n$ about the line $L \in \mathbb{R}^n$ through $\vec{0} + \vec{v} \in \mathbb{R}^n$ is given by:

$$\text{ref}_L(\vec{w}) = 2\,\text{proj}_L(\vec{w}) - \vec{w}$$

$\boxed{\text{Example}}$

$$\text{ref}_L\begin{pmatrix}1\\-1\end{pmatrix} = 2\,\text{proj}_L\begin{pmatrix}1\\-1\end{pmatrix} - \begin{pmatrix}1\\-1\end{pmatrix} = \begin{bmatrix}-2/5\\-6/5\end{bmatrix} - \begin{bmatrix}1\\-1\end{bmatrix} = \begin{bmatrix}-7/5\\-1/5\end{bmatrix}$$

<u>Prop</u> - $\text{ref}_L : \mathbb{R}^n \to \mathbb{R}^n$ is a linear transformation in e.g.

What is $$[\text{ref}_L] = 2[\text{proj}_L] - \underbrace{\begin{bmatrix}1 & 0\\0 & 1\end{bmatrix}}_{\text{Identity matrix}}$$